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String Cosmological Models in Lyra Geometry

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Abstract The Bianchi type-V cosmological solutions of massive strings have been investigated in the theory based on Lyra's geometry in normal gauge, in the presence as well as absence of the magnetic field. The physical and kinematical behaviors of the models have also been discussed.

Keywords Massive string · Lyra geometry · Bianchi type-V model

1 Introduction

The cosmological models which are spatially homogeneous, and anisotropic have a significant role in the description of the universe in the early stages of its evolution. It has been a subject of considerable interest of cosmologist to study alternative theories of gravitation. The most important among them proposed by Lyra [1]. Lyra suggested a modification of Riemannian geometry, which may also be considered as a modification of Weyl's geometry. In Lyra's geometry, Weyl's concept of gauge, which is essentially a metrical concept, is modified by the introduction of a gauge function in to the structure-less manifold. Several authors have studied cosmology in Lyra geometry with a constant as well as time-dependent displacement field, which plays the same role as the cosmological constant in the standard general relativity. Halford [2] studied Robertson-Walker models in Lyra geometry for a time-dependent gauge function. Singh and Shri Ram [3] discussed the spatially homogeneous Bianchi type-I metric in a different basic form and obtained exact solutions of Einstein field equations in vacuum and in the presence of stiff-matter in the normal gauge when the displacement field is time-dependent. Singh and Singh [4] obtained exact solutions for the spatially homogeneous Bianchi type-I model in the normal gauge for Lyra's geometry. Singh and Singh [5] also discussed FRW models in the Lyra manifold with constant deceleration parameter and are relevant to the study of inflationary cosmology.

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The existence of a large-scale network of strings in the early universe is not a contradiction with the present day observations of the universe. These strings posses stress energy and are coupled to gravitational field. Letelier [6] and Stachel [7] studied the gravitational effect of such strings in general relativity. Stachel [7] developed the classical theory of geometric strings as the theory of a simple surface-forming time-like bivector field in an arbitrary background space-time. Letelier [8] obtained relativistic cosmological solutions of cloud formed by massive strings in Bianchi type-I and Kantowsky-Sachs space-times. There is no direct evidence of strings in the present day universe. Accordingly cosmological models of the universe that evolve from a string-dominated era and end up in a particle-dominated era are of physical interest. Matraverse [9] presented a class of exact solutions of Einstein field equations with a two-parameter family of classical strings as the source of the gravitational field. Krori et al. [10] obtained some exact solutions in string cosmology for homogeneous spaces of Bianchi types-II, VI_0, VIII, and IX. Banerjee et al. [11] studied Bianchi type-I strings cosmological models with and without a source-free magnetic field. Tikekar and Patel [12] obtained Bianchi type-III cosmological solutions of massive strings in the presence of magnetic field. Shri Ram and Singh [13] obtained some new exact solutions of string cosmology with and without a source-free magnetic field in the context of Bianchi type-I space-times in the different basic form. Singh and Shri Ram [14] presented a technique to generate new exact Bianchi type-III cosmological solutions of massive strings in the presence and absence of the magnetic field. Singh [15] investigated the Bianchi type-V cosmological solutions of massive strings in the presence and absence of the magnetic field.

In this paper we study the new models of a string cloud and obtain some new exact solutions of the string cosmology with and without a source-free magnetic field in the context of Bianchi type-V space-times in normal gauge for Lyra's geometry. We examine the energy conditions for a cloud of strings coupled to the Einstein equations.

2 Einstein Field Equations

We assume the metric for Bianchi type-V space-time in the general form:

$$ds^2 = dt^2 - e^{2A}dx^2 - e^{2B+2\mu x}dy^2 - e^{2C+2\mu x}dz^2, \tag{1}$$

where the metric potentials A, B, and C are functions of cosmological time t, and μ is the parameter.

The field equations in normal gauge for Lyra's manifold, as obtained by Sen [16] are:

$$R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\left(\phi_i\phi_j - \frac{1}{2}g_{ij}\phi_\alpha\phi^\alpha\right) = -8\pi G T_{ij}. \tag{2}$$

Here ϕ_i is a displacement field vector defined as $\phi_i = (0, 0, 0, \beta)$, where $\beta = \beta(t)$, and other symbols have their usual meaning as in the Riemannian geometry.

The energy-momentum tensor T_{ij} for a cloud of string dust with a magnetic field along the direction of the string is given by

$$T_{ij} = \rho u_i u_j - \lambda w_i w_j + \frac{1}{4\pi}\left(-g^{\alpha\beta}F_{i\alpha}F_{j\beta} + \frac{1}{4}g_{ij}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{3}$$

where ρ is the proper energy density for a cloud of strings with particles attached to them, λ denotes the string tension density, the unit time-like vector u^i describes the cloud four-velocity and the unit space-like vector w^i denotes the direction of the string which can be taken along any one of the three directions $\partial/\partial x, \partial/\partial y, \partial/\partial z$. Without loss of generality let us choose x-direction as the direction of the string along which the magnetic field is assumed to be present. So that $w^i = (e^{-A}, 0, 0, 0)$. In a co-moving coordinate system, we have $u^i = (0, 0, 0, 1)$. Thus we have $u_i u^i = -w^i w_j = 1$, and $u^i w_i = 0$. The electromagnetic field tensor F_{ij} has only one non-zero component F_{23} because the magnetic field is assumed to be along the x-direction. Subsequently Maxwell's equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0,$$

and

$$\left\lfloor F^{ij}\left(\sqrt{-g}\right)\right\rfloor_{,j} = 0 \tag{4}$$

lead to

$$[F^{23}\exp(A + B + C + 2\mu x)]_{,3} = 0,$$
$$\therefore F^{23} = \mathrm{K}, \quad \text{where K is a constant.}$$

Also

$$F_{23} = \mathrm{K}\exp\{2(B + C + 2\mu x)\}. \tag{5}$$

The field equations (2) for the metric (1) lead to the following system of equations:

$$\ddot{B} + \ddot{C} + \dot{B}^2 + \dot{C}^2 + \dot{B}\dot{C} - \mu^2 e^{-2A} = \chi\lambda + \frac{3}{4}\beta^2 - \frac{\chi \mathrm{K}^2}{8\pi} e^{2(B+C+2\mu x)}, \tag{6}$$

$$\ddot{C} + \ddot{A} + \dot{C}^2 + \dot{A}^2 + \dot{C}\dot{A} - \mu^2 e^{-2A} = \frac{3}{4}\beta^2 + \frac{\chi \mathrm{K}^2}{8\pi} e^{2(B+C+2\mu x)}, \tag{7}$$

$$\ddot{A} + \ddot{B} + \dot{A}^2 + \dot{B}^2 + \dot{A}\dot{B} - \mu^2 e^{-2A} = \frac{3}{4}\beta^2 + \frac{\chi \mathrm{K}^2}{8\pi} e^{2(B+C+2\mu x)}, \tag{8}$$

$$\dot{A}\dot{B} + \dot{B}\dot{C} + \dot{C}\dot{A} - 3\mu^2 e^{-2A} = -\chi\rho - \frac{3}{4}\beta^2 - \frac{\chi \mathrm{K}^2}{8\pi} e^{2(B+C+2\mu x)}, \tag{9}$$

$$2\dot{A} - \dot{B} - \dot{C} = 0, \tag{10}$$

where $\chi = 8\pi G$, and a dot denotes differentiation with respect to t. Let ρ_p denotes the particle energy density of the configuration so that

$$\rho = \rho_p + \lambda.$$

The energy conditions lead to $\rho \geq 0$ and $\rho_p \geq 0$, leaving sign of λ unrestricted.

3 Exact Solutions

The field equations (6)–(10) constitute a system of five equations with six unknown parameters A, B, C, ρ, λ, and β. Therefore some additional constraint equations relating these parameters are required to obtain explicit solutions of the system of the equations. From (7)–(8), we get

$$B - C = \ln(t_0 t - t_1)^{2/3}. \tag{11}$$

Assuming $B = mC$, (11) reduces to

$$C = \ln(t_0 t - t_1)^{2/3(m-1)}, \tag{12}$$

where m, t_0 and t_1 are arbitrary constant.

$$A = \ln(t_0 t - t_1)^{(m+1)/3(m+1)}. \tag{13}$$

The physical and kinematical parameters [17] for this model have the following expressions:

$$\lambda = \frac{1}{\chi}\left[\frac{\chi \mathrm{K}^2}{4\pi}(t_0 t - t_1)^{4(m+1)/3(m-1)} \exp(2\mu x) - \frac{2mt_0^2}{3(m-1)(t_0 t - t_1)^2}\right], \tag{14}$$

$$\rho = \frac{1}{\chi}\left[\frac{4\mu^2}{(t_0 t - t_1)^{2(m+1)/3(m-1)}} - \frac{(m^2+6m+7)t_0^2}{3(m-1)^2(t_0 t - t_1)^2}\right], \tag{15}$$

$$\rho_p = \frac{1}{\chi}\left[\frac{4\mu^2}{(t_0 t - t_1)^{2(m+1)/3(m-1)}} + \frac{(m^2-8m-7)t_0^2}{9(m-1)^2(t_0 t - t_1)^2} - \frac{\chi \mathrm{K}^2}{4\pi}(t_0 t - t_1)^{4(m+1)/3(m-1)} \exp(2\mu x)\right], \tag{16}$$

$$\beta^2 = \frac{4}{3}\left[\frac{(m^2+10m+19)t_0^2}{9(m-1)^2(t_0 t - t_1)^2} - \frac{\mu^2}{(t_0 t - t_1)^{2(m+1)/3(m-1)}} - \frac{\chi \mathrm{K}^2}{8\pi}(t_0 t - t_1)^{4(m+1)/3(m-1)} \exp(2\mu x)\right]. \tag{17}$$

Spatial volume:

$$V^3 = (t_0 t - t_1)^{(m+1)/(m-1)} \exp(2\mu x), \tag{18}$$

Expansion scalar:

$$\Theta = \frac{(m+1)t_0}{(m-1)(t_0 t - t_1)}, \tag{19}$$

Shear scalar:

$$\sigma^2 = \frac{(3m^2+2m+3)t_0^2}{9(m-1)^2(t_0 t - t_1)^2}, \tag{20}$$

Hubble parameter:

$$\mathrm{H} = \frac{(m+1)t_0}{3(m-1)(t_0 t - t_1)}, \tag{21}$$

Deceleration parameter:

$$q = \frac{2(m-2)}{(m+1)}, \tag{22}$$

$$\frac{\sigma^2}{\Theta} = \frac{(3m^2+2m+3)t_0}{9(m^2-1)(t_0 t - t_1)}. \tag{23}$$

The relative anisotropy:

$$\frac{\sigma^2}{\rho} = \frac{\chi(3m^2+2m+3)t_0^2}{9(m-1)^2(t_0t-t_1)^2[\frac{4\mu^2}{(t_0t-t_1)^{2(m+1)/3(m-1)}} - \frac{(m^2+6m+7)t_0^2}{3(m-1)^2(t_0t-t_1)^2}]}. \tag{24}$$

It should be noted that the universe exhibits initial singularity of the POINT-type at $t = (t_1/t_0)$. The space-time is well behaved in the range $(t_1/t_0) < t < \infty$. For physical realistic models, we take $t_0, t_1 > 0$ and $m > 1$. At the initial moment $t = (t_1/t_0)$, the physical and kinematical parameters $\rho, \lambda, \rho_p, \beta, \Theta, \sigma^2$, and H tend to infinity and the magnetic field disappeared. So the universe begins from initial singularity with infinite energy density, infinite string tension density, infinite particle energy density, infinitely large gauge function, and with infinite rate of shear and expansion. Moreover $\rho, \lambda, \rho_p, \beta, \Theta, \sigma^2$, and H tend to a finite limit as $t \to 0$. Thus $\rho, \lambda, \rho_p, \beta, \Theta, \sigma^2$, and H are monotonically decreasing for t in the range $(t_1/t_0) < t < \infty$. Near the singularity, the particle will 'dominate' the string ($\rho_p > \lambda$). The expressions (14)–(17) indicate that the magnetic field is linked with λ, ρ_p, and β while it does not effect on ρ. The model gives solution for p-string in presence of the magnetic field. In case the model represents a geometric string model when

$$\begin{aligned} &48\pi\mu^2(m-1)^2(t_0t-t_1)^{4(m-2)/3(m-1)} + 4\pi\left(m^2-8m-7\right)t_0^2 \\ &\quad = 3\chi K^2(m-1)^2(t_0t-t_1)^{2(5m-1)/3(m-1)}\exp(2\mu x). \end{aligned}$$

We also discuss the following properties:

- Near the singularity the string tension density $\lambda \cong 0$ and $\rho \cong \rho_p$ when $2mt_0^2 \cong 0$. At this epoch, the model can be supposed to represent a dust filled universe with magnetic field.
- When $(t_1/t_0) < t < \infty$, the string phase of the universe can disappear because λ may become negative. In this interval, the model can be supposed to contain a highly anisotropic fluid of particles.

The energy conditions $\rho \geq 0$ and $\rho_p \geq 0$ are satisfied when

$$\begin{aligned} &48\pi\mu^2(m-1)^2(t_0t-t_1)^{4(m-2)/3(m-1)} + 4\pi\left(m^2-8m-7\right)t_0^2 \\ &\quad \geq 3\chi K^2(m-1)^2(t_0t-t_1)^{2(5m-1)/3(m-1)}\exp(2\mu x). \end{aligned}$$

The spatial volume V tends to zero at the initial singularity. As time proceeds the universe approaches toward an infinite volume in the limit as $t \to \infty$. For $m > 1$, the expansion scalar Θ is positive, and for $m < 1$, Θ is negative in the interval $(t_1/t_0) < t < \infty$. Therefore the model describes an expanding model for $m > 1$, and a contracting model for $m < 1$ in the presence of magnetic field. The deceleration parameter q is constant but the ratio (σ^2/Θ) of the model is decreasing from a very large quantity to zero in the interval $(t_1/t_0) < t < \infty$. The relative anisotropy (σ^2/ρ) tends to infinity at singularity, and tends to zero as $t \to \infty$. This shows that the model is highly anisotropic at the time of the evolution of the universe. However the model does not admit rotation and acceleration.

In this model particle horizon exists because

$$\int_{t_0}^{t} \frac{dt'}{V(t')} = \left[\frac{3(m-1)(t_0t'-t_1)^{2(m-2)/3(m-1)}}{2(m-2)t_0\exp(2\mu x/3)}\right]_{t_0}^{t}, \quad m > 2 \tag{25}$$

is a convergent integral.

4 Exact Solutions in the Absence of a Magnetic Field

The field equations (6)–(10) in the absence of magnetic field (i.e., K = 0) reduce to

$$\ddot{B}+\ddot{C}+\dot{B}^2+\dot{C}^2+\dot{B}\dot{C}-\mu^2 e^{-2A}=\chi\lambda+\frac{3}{4}\beta^2, \tag{26}$$

$$\ddot{C}+\ddot{A}+\dot{C}^2+\dot{A}^2+\dot{C}\dot{A}-\mu^2 e^{-2A}=\frac{3}{4}\beta^2, \tag{27}$$

$$\ddot{A}+\ddot{B}+\dot{A}^2+\dot{B}^2+\dot{A}\dot{B}-\mu^2 e^{-2A}=\frac{3}{4}\beta^2, \tag{28}$$

$$\dot{A}\dot{B}+\dot{B}\dot{C}+\dot{C}\dot{A}-3\mu^2 e^{-2A}=-\chi\rho-\frac{3}{4}\beta^2, \tag{29}$$

$$\dot{B}-\dot{C}-2\dot{A}=0. \tag{30}$$

To obtain explicit solutions of the system of field equations (26)–(30), we assume $B=\ell C$, where ℓ is an arbitrary constant. The solutions are given by

$$A=\ln(at-b)^{(\ell+1)/3(\ell-1)}, \tag{31}$$

$$B=\ln(at-b)^{2\ell/3(\ell-1)}, \tag{32}$$

$$C=\ln(at-b)^{2/3(\ell-1)}. \tag{33}$$

The physical and kinematical parameters for this model are given by

$$\lambda=\frac{2\ell a^2}{3\chi(1-\ell)(at-b)^2}, \tag{34}$$

$$\rho=\frac{1}{\chi}\left[\frac{4\mu^2}{(at-b)^{2(\ell+1)/3(\ell-1)}}-\frac{(\ell^2+6\ell+7)a^2}{3(\ell-1)^2(at-b)^2}\right], \tag{35}$$

$$\rho_p=\frac{1}{\chi}\left[\frac{4\mu^2}{(at-b)^{2(\ell+1)/3(\ell-1)}}+\frac{(\ell^2-8\ell-7)a^2}{3(\ell-1)^2(at-b)^2}\right], \tag{36}$$

$$\beta^2=\frac{4}{3}\left[\frac{(\ell^2+10\ell+19)a^2}{9(\ell-1)^2(at-b)^2}-\frac{\mu^2}{(at-b)^{2(\ell+1)/3(\ell-1)}}\right]. \tag{37}$$

$$V^3=(at-b)^{(\ell+1)/(\ell-1)}\exp(2\mu x), \tag{38}$$

$$\Theta=\frac{(\ell+1)a}{(\ell-1)(at-b)}, \tag{39}$$

$$\sigma^2=\frac{(3\ell^2+2\ell+3)a^2}{9(\ell-1)^2(at-b)^2}, \tag{40}$$

$$\mathrm{H}=\frac{(\ell+1)a}{3(\ell-1)(at-b)}, \tag{41}$$

$$q=\frac{2(\ell-2)}{(\ell+1)}, \tag{42}$$

$$\frac{\sigma^2}{\Theta}=\frac{(3\ell^2+2\ell+3)a}{9(\ell^2-1)(at-b)}. \tag{43}$$

$$\frac{\sigma^2}{\rho}=\frac{\chi(3\ell^2+2\ell+3)a^2}{9(\ell-1)^2(at-b)^2[\frac{4\mu^2}{(at-b)^{2(\ell+1)/3(\ell-1)}}-\frac{(\ell^2+6\ell+7)a^2}{3(\ell-1)^2(at-b)^2}]}. \tag{44}$$

The universe exhibits initial singularity of the POINT-type at $t=(b/a)$. For physical realistic models, we take $a,b>0$ and $\ell>1$. The space-time is well behaved in the range $(b/a)<t<\infty$. At the initial singularity $t=(b/a)$, the physical and kinematical parameters $\rho,\lambda,\rho_p,\beta,\Theta,\sigma^2$, and H tend to infinity. So the universe starts from initial singularity with infinite energy density, infinite string tension density, infinite particle energy density, infinitely large gauge function, and with infinite rate of shear and expansion. Moreover $\rho,\lambda,\rho_p,\beta,\Theta,\sigma^2$, and H tend to zero as $t\to\infty$. Thus $\rho,\lambda,\rho_p,\beta,\Theta,\sigma^2$, and H are monotonically decreasing toward a zero for t in the interval $(b/a)<t<\infty$. Near the singularity, the particle will 'dominate' the string $(\rho_p>\lambda)$. The model gives solution for p-string in the absence of magnetic field. In case the model represents a cloud of geometric strings when

$$12\mu^2(\ell-1)^2(at-b)^{4(\ell-2)/3(\ell-1)}+(\ell^2-8\ell-7)a^2=0.$$

We also discuss the following properties in the interval $(b/a)<t<\infty$:

- When $t\to t^*=\infty$, the cloud of string tension density λ goes to zero and $\rho=\rho_p$. Thus the model contains only a cloud of particles.
- When $t<t^*$, the string phase of the universe disappears because λ becomes negative, i.e., the model contains a highly anisotropic fluid of particles.
- The critical instant of time $t_c=t^*$ may be calculated by knowing the critical temperature given by GUTs.

The energy conditions $\rho\geq 0$ and $\rho_p\geq 0$ are satisfied when

$$12\mu^2(\ell-1)^2(at-b)^{4(\ell-2)/3(\ell-1)}+(\ell^2-8\ell-7)a^2\geq 0.$$

The spatial volume V tends to zero at the initial singularity. As time proceeds the universe approaches toward an infinite volume in the limit as $t\to\infty$. The expansion scalar Θ is positive for $\ell>1$, and the model undergoes expansion till the epoch $t=\infty$. The expansion scalar Θ is negative for $\ell<1$ consequently there is a contraction till the epoch of singularity $t=(b/a)$. The deceleration parameter q is constant in the interval $(b/a)<t<\infty$. However the ratio (σ^2/Θ) tends to infinity as $t\to(b/a)$, and the ratio (σ^2/Θ) tends to zero as $t\to\infty$. The relative anisotropy (σ^2/ρ) tends to infinity at singularity, and tends to zero at $t\to\infty$. This shows that the model is highly anisotropic at the time of the evolution of the universe in the absence of magnetic field. This model does not admit rotation and acceleration.

In this model particle horizon exists because

$$\int_{t_0}^{t}\frac{dt'}{V(t')}=\left[\frac{3(\ell-1)(at'-b)^{2(\ell-2)/3(\ell-1)}}{2(\ell-2)a\exp(2\mu x/3)}\right]_{t_0}^{t},\quad \ell>2 \tag{45}$$

is a convergent integral.

5 Conclusions

We have studied that both cosmological models evolve with initial singularity of the POINT-type. The universe starts from initial singularity with infinite energy density, infinite string tension density, infinite particle energy density and infinitely large gauge function in the presence as well as absence of a magnetic field. Both models are generated by cloud of strings with particles attached to them. For some particular cases the models of a universe which evolve from a pure geometric string-dominated era on a massive string-dominated era to a particle-dominated with or without a remnant of strings with or without a source free magnetic field. In both cases, the universe begins from initial singularity with infinite rate of shear and expansion. These models describe both expanding and contracting models in certain cases together with constant deceleration parameter.

The models are highly anisotropic at the time of the evolution of the universe. The integrability and the reality conditions for a cloud of massive strings coupled to the Einstein equations in the presence and absence of a magnetic field have also been examined. It is very important to note that a space-time metric with time-dependent displacement vectors based on Lyra's geometry in normal gauge is capable of describing almost all these attributes for suitable values for certain parameters. The change in the parameter values may be accomplished by phase transitions. However both models do not admit rotation and acceleration.

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